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Definition (Heine-Borel) A set $K \subseteq X$ is *compact* if $\forall\, \mathcal{G} \subset \mathcal{T}$ with $\bigcup \mathcal{G} \supset K$, $\exists$ finite $\mathcal{E} \subset \mathcal{G}$ such that $\bigcup \mathcal{E} \supset K$.

In $\mathbb{R}^n$, $A \subset \mathbb{R}^n$ is compact $\Leftrightarrow$ A is closed and bounded

At the same time, A is bounded

$\Leftrightarrow A \subset \{x \in \mathbb{R}^n : |x| \leq R\}$ ← a compact ball

Theorem. Let X be compact and $A \subset X$.

A is closed $\Longrightarrow$ A is compact.

Proof. Let $\mathcal{G} \subset \mathcal{T}$ and $\bigcup \mathcal{G} \supset A$

Since A is closed, $X \setminus A \in \mathcal{T}$

Thus $\mathcal{G} \cup \{X \setminus A\} \subset \mathcal{T}$ and it covers X.

$\exists$ finite $\mathcal{E} \cup \{X \setminus A\}$ covering X, hence $\bigcup \mathcal{E} \supset A$.

Theorem. Let $f: X \to Y$ be continuous and $A \subset X$ be compact. Then $f(A) \subset Y$ is compact.

Proof. Let $\mathcal{G} \subset \mathcal{T}_Y$ with $\bigcup \mathcal{G} \supset f(A)$. Then $\mathcal{G}^* = \{f^{-1}(V) : V \in \mathcal{G}\} \subset \mathcal{T}_X$ and $\bigcup \mathcal{G}^* \supset f^{-1} f(A) \supset A$

By compactness of A, $\exists$ finite $\mathcal{E}^* \subset \mathcal{G}^*$, $\bigcup \mathcal{E}^* \supset A$

The corresponding $\mathcal{E} \subset \mathcal{G}$ when $V \in \mathcal{E} \Leftrightarrow f^{-1}(V) \in \mathcal{E}^*$ is finite and satisfies $\bigcup \mathcal{E} \supset f(A)$.

Consequences

(1) X is compact $\Rightarrow X/\sim = q(X)$ is compact

(2) $P = \prod_{\alpha\in I} X_\alpha$ is compact $\Rightarrow$ each $X_\beta = \pi_\beta(P)$ is so.

Remarks.

* Converse of (1) above is not true, $S^1 = \mathbb{R}/\sim$.
* (2) above is not useful because usually we know each X_β before constructing $P = \prod_{\alpha\in I} X_\alpha$
* Converse of (2) is useful.

Theorem. If each X_α, $\alpha\in I$ is compact then the product $P = \prod_{\alpha\in I} X_\alpha$ is compact.

(i) For finite I, a proof will be given below

(ii) For infinite I, Tychonoff Theorem, only idea.

Proof. Let $(X, \mathcal{T}_X)$, $(Y, \mathcal{T}_Y)$ be compact, and $\mathcal{C} \subset \mathcal{T}_{X\times Y}$ satisfy $\bigcup \mathcal{C} = X\times Y$.

Without loss of generality, assume each set in $\mathcal{C}$ is of the form $U\times V$, $U\in \mathcal{T}_X$, $V\in \mathcal{T}_Y$.

For any fixed $y\in Y$, $X\times\{y\}$ is compact and $\bigcup \mathcal{C} \supset X\times\{y\}$, so it has a finite subcover

$$\mathcal{E}^y = \{U_k^y \times V_k^y : k = 1, \cdots, n_y\}$$

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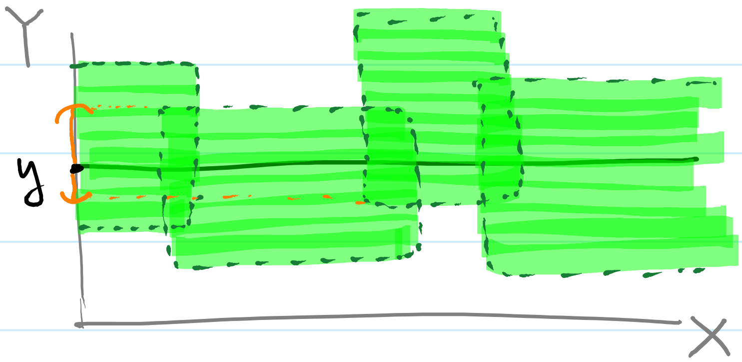
In this case, $\bigcup_{k=1}^{n_y} U_k^y \times V_k^y \supset X \times \{y\}$



As a result, we have

$$\bigcup_{k=1}^{n_y} U_k^y \times V_k^y \supset X \times V^y \supset X \times \{y\} \quad \text{where}$$

$$V^y = \bigcap_{k=1}^{n_y} V_k^y \in \mathcal{T}_Y$$

By this, we have an open cover $\{V^y : y \in Y\}$ for Y and so it has a finite subcover

$$\{V^{y_1}, V^{y_2}, \dots, V^{y_m}\}$$

Then $\mathcal{E} = \{U_k^{y_l} : k = 1, \cdots, n_{y_l}, l = 1, \cdots, m\}$ is a finite subcover for $X \times Y$.

Remark.

1. For general $\mathcal{G} \subset \mathcal{T}_{X \times Y}$, the above finite cover will determine a finite subcover of $\mathcal{G}$.
2. For infinite product, the above may not work, and we look at an equivalent version of compactness.

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Using Contrapositive

Note that $\sim(\bigcup\mathcal{G} \supset X) \Leftrightarrow X \setminus \bigcup\mathcal{G} \neq \phi$

$$X \setminus \bigcup\mathcal{G} = \bigcap\{\underbrace{X\setminus G}_{\text{each is closed, } \mathcal{G}\subset\mathcal{T}_X} : G \in \mathcal{G}\}$$

K is compact $\Longleftrightarrow$

For every $\mathcal{F}$ of closed sets in K, if every finite $\mathcal{H} \subset \mathcal{F}$ satisfies $\bigcap\mathcal{H} \neq \phi$

then $\bigcap\mathcal{F} \neq \phi$

$\Longleftrightarrow$ For every $\mathcal{A} \subset \mathcal{P}(K)$, if $\mathcal{A}$ has FCIP (finite closure intersection property)

i.e., every finite $\mathcal{H} \subset \mathcal{A}$ satisfies

$$\bigcap\overline{\mathcal{H}} = \bigcap\{\overline{H} : H \in \mathcal{H}\} \neq \phi$$

then $\bigcap\overline{\mathcal{A}} \neq \phi$

Idea applied to $X \times Y$.

Let $\mathcal{A} \subset \mathcal{P}(X\times Y)$ have FCIP.

Then both $\mathcal{A}_X = \{\pi_X(A) : A \in \mathcal{A}\} \subset \mathcal{P}(X)$

$\mathcal{A}_Y = \{\pi_Y(A) : A \in \mathcal{A}\} \subset \mathcal{P}(Y)$

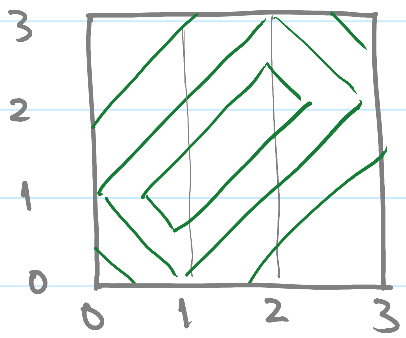
have FCIP. By compactness of X, Y,

$\bigcap \overline{A}_X \neq \phi$, $\bigcap \overline{A}_Y \neq \phi$, i.e., $\exists\ x \in \bigcap \overline{A}_X$, $y \in \bigcap \overline{A}_Y$

Does $(x,y) \in \bigcap \overline{A}$? Unfortunately, not always.

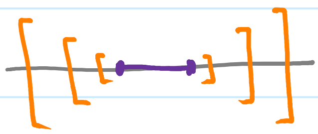
Consider the example $X = Y = [0,3]$ and

$\mathcal{A} \subset \mathcal{P}(X \times Y)$ consists of sets in the picture



Clearly, $\mathcal{A}$ has FCIP

$\mathcal{A}_X$ and $\mathcal{A}_Y$ contain special intervals such that

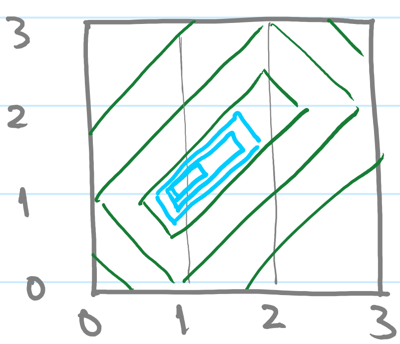


$\bigcap \overline{\mathcal{A}_X} = \bigcap \overline{\mathcal{A}_Y} = [1,2]$

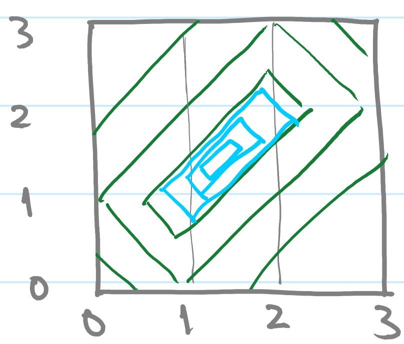
Therefore, according to the proof, we may have $1 \in \bigcap \overline{\mathcal{A}_X}$ and $2 \in \bigcap \overline{\mathcal{A}_Y}$

but $(1,2) \notin \bigcap \overline{\mathcal{A}} \subset$ diagonal

Important Idea is to extend $\mathcal{A}$ to $\mathcal{M}$ so that $\bigcap \overline{\mathcal{M}_X}$, $\bigcap \overline{\mathcal{M}_Y}$ are smaller.



or



$\bigcap \overline{\mathcal{M}_X} = \bigcap \overline{\mathcal{M}_Y} = \{1\}$

$(1,1) \in \bigcap \overline{\mathcal{A}}$

or

$\bigcap \overline{\mathcal{M}_X} = \bigcap \overline{\mathcal{M}_Y} = \{\frac{3}{2}\}$

$(\frac{3}{2}, \frac{3}{2}) \in \bigcap \overline{\mathcal{A}}$

Essential Argument of Tychonoff

$P = \prod_{\alpha \in I} X_\alpha$ is compact

$\iff$ For $\mathcal{A} \subset \mathcal{P}(P)$ having FCIP

From $\mathcal{A}$, by Zorn's Lemma,

create $\mathcal{A} \subset \mathcal{M} \subset \mathcal{P}(P)$

where $\mathcal{M}$ is maximal and

still has FCIP

Then by compactness of X_α,

$x_\alpha \in \bigcap \overline{\mathcal{M}_\alpha}$

The maximality of $\mathcal{M}$ guarantees

$x = (x_\alpha) \in \bigcap \overline{\mathcal{M}}$

Hence, $\bigcap \overline{\mathcal{M}} \neq \emptyset$